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COMPLETIONS OF HANKEL PARTIAL CONTRACTIONS OF SIZE 5×5 NON-EXTREMAL CASE

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Abstract. We introduce a new approach that allows us to solve, algorithmically, the contractive completion problem. In this article, we provide concrete necessary and sufficient conditions for the existence of contractive completions of Hankel partial contractions of size 4×4 using a Moore-Penrose inverse of a matrix.

1. Introduction

A *partial matrix* is a square array in which some entries are specified and others are not. A *completion* of a partial matrix is a choice of values for the unspecified entries. A *matrix completion problem* asks whether a given partial matrix has a completion of a desired type. For example, the positive definite completion problem asks which partial Hermitian matrices have a positive definite completion. For a 2×2 partial operator matrix $A \equiv \begin{pmatrix} B & C \\ D & X \end{pmatrix}$, we say that X is a solution for A, if A is a completion of a desired type. These completion problems have been studied by G. Arsene and A. Gheondea [1], by C. Davis, W. Kahan and H. Weinberger [10] (see also [4] and [9]), by C. Foiaş and A. Frazho [11] (using Redheffer products), by S. Parrott [22], and by Y. L. Shmul'yan and R. N. Yanovskaya [24]; a solution is also implicit in the work of W. Arveson [2] (see also [17] and [23]). A *Hankel matrix* is a square matrix with constant skew-diagonals. A *Toeplitz matrix* is a square matrix in which each descending diagonal from left to right is constant. Hankel and Toeplitz matrices have a long history (see, for instance, [16]) and have given rise to important recent applications in a variety of areas. A matrix completion problem is due, in particular,

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to its many applications, e.g., in probability and statistics (e.g. entropy methods for missing data, see, for instance, [12] and [13]), chemistry (e.g. the molecular conformation problem [5]), numerical analysis (e.g. optimization, see, for instance, [20]), electrical engineering (e.g. data transmission, coding and image enhancement, see, for instance, [3]) and geophysics (seismic reconstruction problems, see, for instance, [14]). A *Hankel Partial Contraction* (HPC) is a square Hankel matrix such that not all of its entries are determined, but in which every well-defined submatrix (completely determined submatrix) is a contraction (in the sense that their operator norms are at most 1). In this article, we study whether a HPC can be completed to a contraction or not when the upper left triangle is known. That is, given real numbers $a_1, \cdots, a_n$, let

(1.1)
$$H_n \equiv H_n(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1}) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix}$$

be a Hankel matrix, where $x_1, \cdots, x_{n-1}$ are real numbers to be determined. We then consider:

Problem 1.1. *Find the necessary and sufficient conditions on the given real numbers $a_1, a_2, \cdots, a_n$ as in (1.1) to guarantee the existence of a contractive Hankel completion.*

We say that Problem 1.1 is *well-posed* if H_n is partially contractive, and that it is *soluble* if H_n is contractive for some $x_1, \cdots, x_{n-1}$. We also say that H_n is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$.

In [7, Section 4], R. Curto, S. Lee and J. Yoon found necessary and sufficient conditions for the existence of contractive completion of HPC's of the extremal type for 4×4 matrices. In this paper, we improve and extend the main results in [7, Section 4] to the non-extremal type for 4×4 matrices and extremal type for 5×5 matrices, respectively. We also give a negative answer to the conjecture presented in [7, Remark 4.5]. We find concrete necessary and sufficient conditions for the existence of completion of 4×4 and 5×5 Hankel partial contractions using the Nested Determinants Test (or Choleski's Algorithm), the Moore-Penrose inverse of a matrix, the Schur product techniques of matrices, and the congruence of the positivity for two matrices. All these techniques may allow us to solve, algorithmically, the contractive completion problem for the non-extremal type of 5×5 Hankel matrices and more.

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2. Preliminaries

For the reader's convenience, in this section, we gathered several auxiliary results which are needed for the proofs of the main results in this article. We first recall that an $n \times n$ matrix $M_{n\times n}$ is a contraction if and only if the matrix

$$P_n \equiv P_n(M_{n\times n}) := I - M_{n\times n}M_{n\times n}^*$$

is positive semi-definite (in symbols, $P_n \geq 0$), where I is the identity matrix and $M_{n\times n}^*$ is the adjoint of $M_{n\times n}$. In order to check the positivity of P_n, we use the following version of the Nested Determinants Test.

LEMMA 2.1. ([6]) *Assume*

$$P \equiv (p_{ij})_{i,j=1}^n := \begin{pmatrix} u & \mathbf{t} \\ \mathbf{t}^* & P_0 \end{pmatrix},$$

where P_0 is an $(n-1)\times(n-1)$ matrix, $\mathbf{t}$ is a row vector, and u is a real number.

(i) *If P_0 is invertible, then* $\det P = \det P_0(u - \mathbf{t}P_0^{-1}\mathbf{t}^*)$.
(ii) *If P_0 is invertible and positive, then* $P \geq 0 \iff (u - \mathbf{t}P_0^{-1}\mathbf{t}^*) \geq 0 \iff \det P \geq 0$.
(iii) *If $u > 0$ then* $P \geq 0 \iff P_0 - \mathbf{t}^*u^{-1}\mathbf{t} \geq 0$.
(iv) If $P \geq 0$ and $p_{ii} = 0$ for some i, $1 \leq i \leq n$, then $p_{ij} = p_{ji} = 0$ for all $j = 1, \cdots, n$.

We next consider:

LEMMA 2.2. (cf. [25]) *Let* $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then,*

$$M \geq 0 \text{ if and only if there exists } W \text{ such that } \begin{cases} A \geq 0, \\ B = AW, \text{ and} \\ C \geq W^*AW. \end{cases}$$

For a $m \times n$ matrix A, the *Moore-Penrose inverse* of A is defined as a matrix as a $n \times m$ matrix $A^\dagger$ satisfying all of the following four conditions:

(i) $AA^{\dagger}A = A$; (ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$; (iii) $\left(AA^{\dagger}\right)^{*} = AA^{\dagger}$; (iv) $\left(A^{\dagger}A\right)^{*} = A^{\dagger}A$.

The following result is a variant of Lemma 2.2.

LEMMA 2.3. ([8, Lemma 1.2]) *Let* $M \equiv \begin{pmatrix} A & B \\ B^{*} & C \end{pmatrix}$ *be a finite matrix. Then,* $M \geq 0$ *if and only if the following three conditions hold:*

(i) $A \geq 0$;
(ii) $\mathrm{ran}B \subseteq \mathrm{ran}\ A$; *and*
(iii) $C \geq B^{*}A^{\dagger}B$, *where* $A^{\dagger}$ *is the Moore-Penrose inverse of* A.

The following result suggests that we could complete $\begin{pmatrix} A & B \\ C & * \end{pmatrix}$ as a contraction provided that $\begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} A & C \end{pmatrix}^{T}$ are contractive. This Lemma makes some contribution to establish main results.

LEMMA 2.4. *(cf.* [10], [22]*) If* $\begin{pmatrix} A & B \end{pmatrix}$ *and* $\begin{pmatrix} A & C \end{pmatrix}^{T}$ *are contractions, then there exists a matrix* D *such that the matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a contraction as well.*

Here, we pose to introduce matrices whose positive semi-definiteness and determinant play an important role in getting main results. For $-1 \leq a, b, c, d \leq 1$, we let

$$H_{22}(x) := \begin{pmatrix} a & b \\ b & x \end{pmatrix},\ H_{23}(x) := \begin{pmatrix} a & b & c \\ b & c & x \end{pmatrix},\ H_{24}(x) := \begin{pmatrix} a & b & c & d \\ b & c & d & x \end{pmatrix},$$

$$H_{33}(x) := \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & x \end{pmatrix},\ H_{32} := \begin{pmatrix} b & c \\ c & d \\ d & e \end{pmatrix}$$

and define a matrix-valued function $P(A) := I - AA^{*}$, where I is the identity matrix of the same size as AA^{*}. We also let

$$P_{22}(x) := P(H_{22}(x)), \qquad P_{23}(x) := P(H_{23}(x)),$$
$$P_{24}(x) := P(H_{24}(x)), \qquad P_{33}(x) := P(H_{33}(x)), \quad \text{and} \quad R_{23} := P(H_{32}).$$

Then, we investigate some connections among the matrices given above:

LEMMA 2.5. *If* $1-a^2-b^2 > 0$ *and* $\det P_{22}(c) = 0$, *then* $\det P_{23}(x) \geq 0$ *if and only if* $x = -\frac{bc(a+c)}{1-a^2-b^2}$.

COROLLARY 2.6. *If* $1 - a^2 - b^2 - c^2 > 0$, *then the following holds:*

(i) *For some* $-1 \leq x \leq 1$, *if* $\det P_{23}(x) = 0$, *then* $\det P_{22}(c) \geq 0$;
(ii) *For some* $-1 \leq x \leq 1$, $\det P_{23}(x) > 0$ *if and only if* $\det P_{22}(c) > 0$;

(iii) *If* $\det P_{22}(c) = 0$, *then there is* x *such that* $\det P_{23}(x) = 0$. *Indeed,* $x = -\frac{bc(a+c)}{1-a^2-b^2}$.

For the next auxiliary results, we observe by Lemma 2.3 (iii) that

$$H_{33}(x) \text{ is a HPC} \iff f_1(x) := \alpha_1 x^2 + \beta_1 x + \gamma_1 \geq 0;$$
$$H_{24}(x) \text{ is a HPC} \iff f_2(x) := \alpha_2 x^2 + \beta_2 x + \gamma_2 \geq 0,$$

where the coefficients are

$$\alpha_1 := -\tfrac{\det P_{22}(c)}{1-a^2-b^2-c^2},$$

$$\beta_1 := \tfrac{2\left(-ac^2-b^2c^3+ac^4-2bcd+2b^3cd-2abc^2d-ab^2d^2-cd^2+a^2cd^2\right)}{1-a^2-b^2-c^2},$$

$$\gamma_1 := \tfrac{\left(1-b^2+ac-c^2-d^2\right)^2-\left(a+c-c^3+2bcd-ad^2\right)^2}{1-a^2-b^2-c^2},$$

$\alpha_2 := -\left(1-a^2-b^2-c^2\right)$, $\beta_2 := -2d\,(ab+bc+cd)$, and

$$\gamma_2 := \det P_{23}(d) - d^2\left(1-b^2-c^2-d^2\right).$$

Let $\mathcal{S}_+(i)$ (resp. $\mathcal{S}_-(i)$) $\subseteq [-1,1]$ be the solution set of $f_i(x) \geq 0$ (resp. $f_i(x) < 0$) for $i = 1, 2$. Then, we inspect how $\det P_{23}(d)$ affects on the solution set of the quadratic inequalities of $f_i(x) \geq 0$:

LEMMA 2.7. *If* $1-a^2-b^2-c^2-d^2 > 0$, *then the following statements are true:*

(i) *The discriminant* $\beta_1^2 - 4\alpha_1\gamma_1$ *of the quadratic equation* $f_1(x)$ *is*

$$\beta_1^2 - 4\alpha_1\gamma_1 = \frac{4\left(\det P_{23}(d)\right)^2}{\left(1-a^2-b^2-c^2\right)^2};$$

(ii) *The discriminant* $\beta_2^2 - 4\alpha_2\gamma_2$ *of the quadratic equation* $f_2(x)$ *is*

$$\beta_2^2 - 4\alpha_2\gamma_2 = 4\left(1-a^2-b^2-c^2-d^2\right)\det P_{23}(d).$$

Furthermore, if $1-a^2-b^2-c^2-d^2 > 0$, then for $i = 1, 2$, we have

$$\mathcal{S}_+(i) \neq \varnothing \Longrightarrow \det P_{23}(d) \geq 0. \tag{2.1}$$

We conclude this section with a helpful tool used in the proof of main results:

LEMMA 2.8. *Problem 1.1 is soluble for* $H_4 \equiv H_4(a,b,c,d;x,y,z)$ *if and only if there exists* x *satisfying both inequalities*

$$\|H_{24}(x)\| \leq 1 \qquad \textit{and} \qquad \|H_{33}(x)\| \leq 1. \tag{2.2}$$

3. Partially contractive Hankel matrices of extremal type: The case 4×4

Since $\|S\| \leq \|T\|$, if S is a submatrix of the matrix T with $\|T\| \leq 1$, then S is again a contraction. Thus, a necessary condition for a partial matrix T to be a contraction is that each submatrix must be a contraction. We call a partial matrix meeting this necessary condition a *partial contraction* (or *well-posed condition*).

In this section, we improve the main results in [7]; Theorem 3.2 given below covers the results in [7, Theorems 4.2, 4.3 and 4.4] at a time. We need to introduce another matrices to establish the main results; let

$$H_{34}(x,y) := \begin{pmatrix} a & b & c & d \\ b & c & d & x \\ c & d & x & y \end{pmatrix} \qquad \text{and} \qquad H_{43}(x,y) := \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & x \\ d & x & y \end{pmatrix}.$$

Also, let $C_1(a,b,c,d) := (a+c)(b+d)(ad-ab-bc-cd)$ and $C_2(a,b,c,d) := |(a+c)(b+d)| - |ac+bd|$. We next present more concrete conditions for the solubility of H_4 according to the values of d:

THEOREM 3.1. ([7]) *Assume $d = 0$. Then, Problem 1.1 is soluble for H_4 if and only if*

$$b(a+c) = 0.$$

We also have:

THEOREM 3.2. ([19]) *Assume $d \neq 0$. Then, Problem 1.1 is soluble for H_4 if and only if the following two conditions hold:*

(i) $C_1(a,b,c,d) \geq 0$ *and*
(ii) $C_2(a,b,c,d) \geq 0$.

4. Partially contractive Hankel matrices of non-extremal type: The case 4×4

We now pay attention to the non-extremal type for 4×4 Hankel matrices of the form $H_4 \equiv H_4(a,b,c,d;x,y,z)$ (that is, when $a^2+b^2+c^2+d^2 < 1$). Consider the solubility of Problem 1.1 for a Hankel matrix H_4 which is well-posed. For $i = 1, 2$, suppose $\alpha_i < 0$ and β_i, $\gamma_i \in \mathbb{R}$. We recall that $\mathcal{S}_+(i)$ (resp. $\mathcal{S}_-(i)$) $\subseteq [-1,1]$ is the solution

set of the quadratic inequality equation $f_i(x) = \alpha_i x^2 + \beta_i x + \gamma_i \geq 0$ (resp. $f_i(x) < 0$). We next let

$$k_1 := \frac{-c(ac+bd)-\left(1-a^2-b^2-c^2\right)}{1-a^2-b^2} \text{ and } k_2 := \frac{-c(ac+bd)+\left(1-a^2-b^2-c^2\right)}{1-a^2-b^2}.$$

THEOREM 4.1. ([19]) *Assume* $\det P_{22}(c) = 0$. *Then Problem 1.1 is soluble for* H_4 *if and only if*

$$k_1 \leq \frac{2b^2c\,(a+c)^2}{\left(1-a^2-b^2\right)^2} \leq k_2.$$

We next have:

THEOREM 4.2. ([19]) *Assume* $\det P_{22}(c) > 0$. *Then, Problem 1.1 is soluble for* H_4 *if and only if*

$$\mathcal{S}_+(1) \cap \mathcal{S}_+(2) \neq \varnothing.$$

5. Partially contractive Hankel matrices of extremal type: The case 5×5

In this section, we focus on the extremal case for $H_5 = H(a, b, c, d, e; x, y, z, w)$. Our approach requires that we split the analysis into two cases ($e = 0$ and $e > 0$), because we get a similar result using the repeated calculations in the proofs of Theorems 5.2, 5.3, 5.4, and 5.5 given below for the case $e < 0$. Consider the solubility of Problem 1.1 for a Hankel matrix H_5, which is well-posed. By Lemma 2.4 and Lemma 2.8, we first observe that Problem 1.1 is soluble for H_5 if and only if there exist x and y such that we simultaneously have

(5.1)

$$\|H_{25}(x)\| \leq 1,\ \|H_{34}(x)\| \leq 1,\ \|H_{35}(x,y)\| \leq 1, \text{ and } \|H_{44}(x,y)\| \leq 1,$$

where

$$H_{25}(x) := \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & x \end{pmatrix},\ H_{34}(x) := \begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & x \end{pmatrix},$$

$$H_{35}(x,y) := \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & x \\ c & d & e & x & y \end{pmatrix}, \text{ and } H_{44}(x,y) := \begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & x \\ d & e & x & y \end{pmatrix}.$$

To obtain our results, we let

$P_{25}(x) := P(H_{25}(x))$, $P_{34}(x) := P(H_{34}(x))$, $P_{35}(x) := P(H_{35}(x))$, and $P_{44}(x) := P(H_{44}(x))$.

We have relied heavily on the Nested Determinant Test so far for checking positivity of matrices; however, we need to use different approaches in this section: For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, where $(A \circ B)_{i,j} := (A)_{i,j}(B)_{i,j}$ for $1 \leq i, j \leq n$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ [21]. Recall that two matrices $A, B \in M_n(\mathbb{R})$ are called *congruent* if there exists an invertible matrix $Q \in M_n(\mathbb{R})$ such that $B = Q^T A Q$. The following result is also well known: $A \geq 0 \iff Q^T A Q \geq 0$ [15]; the facts will be used to prove Theorem 5.5.

We are ready to consider the first case:

The case $e = 0$. Using the Nested Determinants Test in Lemma 2.1 and eliminating the common factors in matices $P_{25}(x)$, $P_{34}(x)$, $P_{35}(x, y)$ and $P_{44}(x, y)$, respectively, we can show

$$\|H_{25}(x)\| \leq 1 \iff \{|x| \leq |a| \text{ and } ab + bc + cd = 0, \tag{5.2}$$

$$\|H_{34}(x)\| \leq 1 \iff \begin{cases} ab + bc + cd = ac + bd + dx = 0 \text{ and} \\ A(x) := \begin{pmatrix} a^2 & ab \\ ab & a^2 + b^2 - x^2 \end{pmatrix} \geq 0, \end{cases} \tag{5.3}$$

(5.4)

$$\|H_{35}(x, y)\| \leq 1 \iff \begin{cases} ab + bc + cd = ac + bd + dx = 0 \text{ and} \\ B(x, y) := \begin{pmatrix} a^2 - x^2 & ab - xy \\ ab - xy & a^2 + b^2 - x^2 - y^2 \end{pmatrix} \geq 0, \end{cases}$$

and

(5.5)

$$\|H_{44}(x, y)\| \leq 1$$

$$\iff \begin{cases} ac + bd + dx = ad + cx + dy = 0 \text{ and} \\ C(x, y) := \begin{pmatrix} a^2 & -c(b+d) & ac \\ -c(b+d) & a^2 + b^2 - x^2 & -cd - xy \\ ac & -cd - xy & 1 - d^2 - x^2 - y^2 \end{pmatrix} \geq 0. \end{cases}$$

Then, we have:

THEOREM 5.1. ([19]) *Assume $e = 0$. Then, Problem 1.1 is soluble for H_5 if and only if one of the following hold:*

(i) $d = 0$ *and* $ac = b(a + c) = 0$;
(ii) $d \neq 0$, $ab + bc + cd = 0$, *and* $|ac + bd| \leq |ad|$.

The case $e > 0$**.** Direct calculations (i.e., the Nested Determinants Test in Lemma 2.1 and eliminating the common factors in matices $P_{25}(x)$, $P_{34}(x)$, $P_{35}(x, y)$ and $P_{44}(x, y)$) imply

$$\|H_{25}(x)\| \leq 1 \iff x = -\tfrac{ab+bc+cd+de}{e} \text{ and } |x| \leq |a|, \tag{5.6}$$

$$\|H_{34}(x)\| \leq 1$$

$$\iff \begin{pmatrix} \frac{a^2e^2-(ab+bc+cd+de)^2}{e^2} & m(x) \\ m(x) & \frac{\left(a^2+b^2-x^2\right)e^2-(ac+bd+ce+dx)^2}{e^2} \end{pmatrix} \geq 0, \tag{5.7}$$

$$\|H_{35}(x, y)\| \leq 1 \tag{5.8}$$

$$\iff \begin{cases} x = -\frac{ab+bc+cd+de}{e},\ y = \frac{abd+bcd+cd^2+d^2e-ace-bde-ce^2}{e^2}, \text{ and} \\ \begin{pmatrix} a^2 - x^2 & ab - xy \\ ab - xy & a^2 + b^2 - x^2 - y^2 \end{pmatrix} \geq 0, \end{cases}$$

and

$$\|H_{44}(x, y)\| \leq 1 \iff M := \begin{pmatrix} f(x) & g(x, y) & h(x, y) \\ g(x, y) & j(x, y) & k(x, y) \\ h(x, y) & k(x, y) & \ell(x, y) \end{pmatrix} \geq 0, \tag{5.9}$$

where

$$m(x) := \tfrac{e^2(bc+cd+de+ex)-(ab+bc+cd+de)(ac+bd+ce+dx)}{e^2},$$

$$f(x) := a^2 - x^2,\ g(x, y) := ab - xy,\ j(x, y) := a^2 + b^2 - x^2 - y^2$$

$$h(x, y) := \tfrac{ace+adx+bex+dxy}{e},\ k(x, y) := \tfrac{abe+bce+ady+bey+cxy-exy+dy^2}{e}, \text{ and}$$

$$\ell(x, y) := \tfrac{a^2e^2-a^2d^2+c^2e^2-abde-2c(ad+be)x-\left(c^2+e^2\right)x^2-2d(ad+be+cx)y-\left(d^2+e^2\right)y^2}{e^2}.$$

Then, we have:

THEOREM 5.2. ([19]) *Assume* $e > 0$ *and* $a = 0$. *Then, Problem 1.1 is soluble for* H_5 *if and only if the following two conditions hold:*

(i) $bc + cd + de = 0$ *and*

(ii) $|bd + ce| \leq |be|$.

We let $s := ab + bc + cd + de + ea$. Then, we have:

THEOREM 5.3. ([19]) *Assume $e > 0$, $a \neq 0$, and $s = 0$. Then, Problem 1.1 is soluble for H_5 if and only if the following three conditions hold:*

(i) $a + d \neq 0$;
(ii) $b = c$;
(iii) $ab + bd + da = 0$.

THEOREM 5.4. ([19]) *Assume $e > 0$, $a \neq 0$, and $s = 2ea$. Then, Problem 1.1 is soluble for H_5 if and only if one of the following three conditions hold:*

(i) $a - d \neq 0$;
(ii) $b = 0$;
(iii) $ad = c\,(a + e)$.

For the next result, let $t := ac + ad + bd + be + ce$ and $v := ac - ad + 2ae + bd - be + ce - s$. We also let $w_1\,(s) := s^2 - (ad + 2ae + be)s + aet$ and $w_2\,(s) := s^2 + (ad - 2ae + be)s - aet$. Then, we have:

THEOREM 5.5. ([19]) *Assume $e > 0$, $a \neq 0$, $s \neq 0$, and $s \neq 2ea$. Then, Problem 1.1 is soluble for H_5 if and only if the following three conditions hold:*

(i) $s\,(2ae - s) > 0$
(ii) $w_1\,(s)\,w_2\,(s) \geq 0$;
(iii) $v(s + t) \geq 0$.

6. Partially contractive Hankel matrices of non-extremal type: The case 5×5

We now focus our attention on the non-extremal case for 5×5 Hankel matrices, i.e., $a^2 + b^2 + c^2 + d^2 + e^2 < 1$. Let $\alpha_5 := -(1 - a^2 - b^2 - c^2 - d^2)$, $\alpha_6 := \frac{-\det P_{23} x^2}{1-a^2-b^2-c^2-d^2}$, $\beta_5 := -2e(ab + bc + cd + de)$, $\beta_6 := \frac{2d(ac+bd+ce)\left(1-b^2-c^2-d^2-e^2\right)}{1-a^2-b^2-c^2-d^2} + 2abe$, $\gamma_5 := -e^2(1 - b^2 - c^2 - d^2 - e^2) + \det P_{24}$ and $\gamma_6 := \frac{(ac+bd+ce)^2\left(1-b^2-c^2-d^2-e^2\right)}{1-a^2-b^2-c^2-d^2} + b^2 \det P_{23}$. We also let $\alpha_7 := -e\,(ab + bc + cd + de)$, $\beta_7 := -2d\,(ac + bd + ce)$ and $\gamma_7 := \left(1 - a^2 - b^2 - c^2 - d^2\right)\left(1 - c^2 - d^2 - e^2\right) - (ac + bd + ce)^2$. Then we have

THEOREM 6.1. *Let $\rho := ab + bc + cd + de = 0$. Then Problem 1.1 is soluble for H_5 if and only if $\mathcal{S}_+(5) \cap \mathcal{S}_+(6) \neq \varnothing$.*

Proof. Since $ab + bc + cd + de = 0$, by Lemma [25] and a direct calculation, we have

(6.1)

$$\|H_{25}(x)\| \leq 1$$

$$\iff \begin{pmatrix} 1-a^2-b^2-c^2-d^2-e^2 & -ab-bc-cd-de-ex \\ -ab-bc-cd-de-ex & 1-b^2-c^2-d^2-e^2-x^2 \end{pmatrix} \geq 0$$

$$\iff \alpha_5 x^2 + \beta_5 x + \gamma_5 \geq 0$$
$$\iff x_5 \leq x \leq x_6,$$

and

(6.2)

$$\|H_{34}(x)\| \leq 1$$

$$\iff \begin{pmatrix} 1-a^2-b^2-c^2-d^2 & -ab-bc-cd-de & -ac-bd-ce-dx \\ -ab-bc-cd-de & 1-b^2-c^2-d^2-e^2 & -bc-cd-de-xe \\ -ac-bd-ce-dx & -bc-cd-de-xe & 1-c^2-d^2-e^2-x^2 \end{pmatrix} \geq 0$$

$$\iff \begin{pmatrix} 1-b^2-c^2-d^2-e^2 & -bc-cd-de-xe \\ -bc-cd-de-xe & 1-c^2-d^2-e^2-x^2-\frac{(ac+bd+ce+dx)^2}{1-a^2-b^2-c^2-d^2} \end{pmatrix} \geq 0$$

$$\iff \alpha_6 x^2 + \beta_6 x + \gamma_6 \geq 0$$

$$\iff \frac{-\beta_6 - \sqrt{\beta_6^2 - 4\alpha_6\gamma_6}}{\alpha_6} \leq x \leq \frac{-\beta_6 + \sqrt{\beta_6^2 - 4\alpha_6\gamma_6}}{\alpha_6}.$$

where, $x_5 := \frac{-e(ab+bc+cd+de) - \sqrt{(1-a^2-b^2-c^2-d^2-e^2)\det P_{24}}}{1-a^2-b^2-c^2-d^2}$ and

$x_6 := \frac{-e(ab+bc+cd+de) + \sqrt{(1-a^2-b^2-c^2-d^2-e^2)\det P_{24}}}{1-a^2-b^2-c^2-d^2}$. Observe that $\beta_6^2 - 4\alpha_6\gamma_6 \geq 0$. Thus, by the above analysis, we have that Problem 1.1 is soluble for H_4 if and only if $\mathcal{S}_+(5) \cap \mathcal{S}_+(6) \neq \varnothing$. □

Let $\sigma := 1 - a^2 - b^2 - c^2 - d^2$ and $\tau := ac + bd + ce$. Then $\det P_{24} = \rho^2 - \sigma\left(\sigma + a^2 - e^2\right)$, so we have

THEOREM 6.2. *Let $\rho \neq 0$ and $\det P_{24} = 0$. Then Problem 1.1 is soluble for H_5 if and only if the following three conditions hold:*

(i) $de\rho^2 - \rho\sigma\tau + e^2\rho\sigma + ab\sigma^2 - \rho\sigma^2 = 0$ *and*
(ii) $x = \frac{-e\rho}{\sigma} \in \mathcal{S}_+(7)$.

Proof. We recall that

$$\|H_{25}(x)\| \le 1 \Longleftrightarrow x = \frac{-e\rho}{\sigma} \tag{6.3}$$

and

$$\|H_{34}(x)\| \le 1 \Longleftrightarrow \begin{pmatrix} 0 & ab-\rho-xe-\frac{\rho(\tau+dx)}{\sigma} \\ ab-\rho-xe-\frac{\rho(\tau+dx)}{\sigma} & 1-c^2-d^2-e^2-x^2-\frac{(\tau+dx)^2}{\sigma} \end{pmatrix} \ge 0 \tag{6.4}$$

Let $\alpha_7 := -\left(\sigma - d^2\right)$, $\beta_7 := -2d\tau$ and $\gamma_7 := \left(1-c^2-d^2-e^2\right)\sigma + \tau^2$. Then we note that $\beta_7^2 \ge 4\alpha_7\gamma_7$. Thus, by (6.4), we have

$$\begin{aligned} &\|H_{34}(x)\| \le 1 \Longleftrightarrow \alpha_7 x^2 + \beta_7 x + \gamma_7 \ge 0 \\ &\Longleftrightarrow \frac{-\beta_7-\sqrt{\beta_7^2-4\alpha_7\gamma_7}}{\alpha_7} \le x \le \frac{-\beta_7+\sqrt{\beta_7^2-4\alpha_7\gamma_7}}{\alpha_7}. \end{aligned}$$

Therefore, by (6.3) and (6.4), we have

$$de\rho^2 - \rho\sigma\tau + e^2\rho\sigma + ab\sigma^2 - \rho\sigma^2 = 0$$

and $x = \frac{-e\rho}{\sigma} \in \mathcal{S}_+(7)$, as desired. □

THEOREM 6.3. *$\rho \ne 0$ and $\det P_{24} \ne 0$. Then Problem 1.1 is soluble for H_5 if and only if the following two conditions hold:*

(i) $\beta_8^2 - 4\alpha_8\gamma_8 \ge 0$ *and*
(ii) $\mathcal{S}_+(5) \cap \mathcal{S}_+(8) \ne \varnothing$.

Proof. Note that

$$\|H_{25}(x)\| \le 1 \Longleftrightarrow \mathcal{S}_+(5) \ne \varnothing \tag{6.5}$$

and

$$\|H_{34}(x)\| \le 1 \Longleftrightarrow \begin{pmatrix} \sigma+a^2-e^2-\frac{\rho^2}{\sigma} & ab-\rho-xe-\frac{\rho(\tau+dx)}{\sigma} \\ ab-\rho-xe-\frac{\rho(\tau+dx)}{\sigma} & 1-c^2-d^2-e^2-x^2-\frac{(\tau+dx)^2}{\sigma} \end{pmatrix} \ge 0 \tag{6.6}$$

Let $\alpha_8 := \frac{\left(a^2+d^2+\sigma\right)\sigma - a^2d^2 - (de-\rho)^2}{\sigma}$,
$\beta_8 := \frac{2\left(abd\rho - d\rho^2 + abe\sigma - e\rho\sigma - a^2d\tau + de^2\tau - e\rho\tau - d\sigma\tau\right)}{\sigma}$, and
$\gamma_8 := \left[\sigma\left(\sigma+a^2-e^2\right)\right]\left[\sigma\left(1-c^2-d^2-e^2\right)-\tau^2\right] - \left[\sigma(ab-\rho)-\rho\tau\right]$.
Thus, by (6.6), we have

$$\begin{aligned} &\|H_{34}(x)\| \le 1 \Longleftrightarrow \alpha_8 x^2 + \beta_8 x + \gamma_8 \ge 0 \\ &\Longleftrightarrow \frac{-\beta_8-\sqrt{\beta_8^2-4\alpha_8\gamma_8}}{\alpha_8} \le x \le \frac{-\beta_8+\sqrt{\beta_8^2-4\alpha_8\gamma_8}}{\alpha_8}, \text{ if } \beta_8^2 - 4\alpha_8\gamma_8 \ge 0 \end{aligned}$$

Therefore, by (6.3) and (6.4), we have that Problem 1.1 is soluble for H_5 if and only if $\mathcal{S}_+(5) \cap \mathcal{S}_+(8) \neq \varnothing$, if $\beta_8^2 - 4\alpha_8\gamma_8 \geq 0$. □

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